

THE BHATTACHARYA FUNCTION OF COMPLETE MONOMIAL IDEALS IN TWO VARIABLES

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ABSTRACT. We give explicit formulas for the Bhattacharya function of \mathfrak{m} -primary complete monomial ideals in two variables in terms of the vertices of the Newton polyhedra or in terms of the decompositions of the ideals as products of simple ideals.

INTRODUCTION

Let (R, \mathfrak{m}) be a d -dimensional local ring. For two \mathfrak{m} -primary ideals I, J of R , one can consider the length function $\ell(R/I^m J^n)$, $m, n \geq 0$. This function was studied first by Bhattacharya [1] who proved that this function is a polynomial $P(m, n)$ of total degree d for $m, n \gg 0$. One calls $\ell(R/I^m J^n)$ and $P(m, n)$ the Bhattacharya function and the Bhattacharya polynomial of I, J . If we write

$$P(m, n) = \sum_{0 \leq i, j \leq d} e_{i,j} \binom{m}{i} \binom{n}{j},$$

then the numbers $e_{i,j}$ with $i + j = d$ are called the mixed multiplicities of I, J . In general, it is very difficult to compute the Bhattacharya polynomial or the mixed multiplicities of a given pair of ideals I, J .

If I and J are \mathfrak{m} -primary monomial ideals of a polynomial ring R over a field, where \mathfrak{m} is the maximal homogeneous ideal, Teissier [8, Appendix] showed that the mixed multiplicities can be interpreted in terms of the volumes of the complements of the Newton polyhedra of I , and J . Recall that the Newton polyhedron of a monomial ideal is the convex hull of the exponent vectors of the monomials of the ideal. But nothing is known about the other coefficients of the Bhattacharya polynomial in this general setting.

The only case where one can compute the Bhattacharya polynomial is the case (R, \mathfrak{m}) is a two-dimensional regular local ring and I, J are \mathfrak{m} -primary complete ideals. Recall that an ideal is called complete if it is integrally closed. Using joint reductions of complete ideals and Lipman's formula for mixed multiplicities, Verma [11] showed that the Bhattacharya function coincides with the Bhattacharya polynomial and

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that its coefficients can be expressed in terms of the multiplicities and the lengths of R/I , R/J , and R/IJ .

In this paper we will compute the Bhattacharya polynomial in the case R is a polynomial ring in two variables and I, J are \mathfrak{m} -primary complete monomial ideals. Our results show that the coefficients of the Bhattacharya polynomial can be explicitly expressed in terms of the vertices of Newton polyhedra of I and J or in terms of the decompositions of I, J as products of simple ideals. Recall that a complete ideal is called simple if it is not the product of two proper complete ideals.

Our approach is based on a result of Crispin Quinonez [2] who described the decomposition of an \mathfrak{m} -primary complete monomial ideal as a product of simple ideals in terms of the vertices of the Newton polyhedron. The existence of such a decomposition follows from Zariski's theory of complete ideals in a two-dimensional regular local ring [13, Appendix 5]. The description of such a decomposition allows us to study the combinatorial properties of the product of two complete monomial ideals. We will reduce the problem of computing the Bhattacharya function to the problem of counting lattice points of the complement of the Newton polyhedron and we will use Pick's theorem to relate the number of lattice points to the area of the corresponding polygon. If we know the decomposition of the ideal as a product of simple ideals, we can compute the area of such a polygon by the theory of Minkowski sum and mixed areas.

Our results do not have any theoretical contribution to the theory of Bhattacharya function and mixed multiplicities. However, we feel that combinatorial formulas for the Bhattacharya function of complete monomial ideals in two variables would be useful for the study of mixed multiplicities and similar functions of monomial ideals in several variables. For instance, if I is an \mathfrak{m} -primary monomial ideal and J an arbitrary monomial ideal, one can use the function $\ell(I^n J^m / I^{n+1} J^m)$ to define mixed multiplicities of I, J [1], [6], [10]. So far, no combinatorial formula for these mixed multiplicities is known except the case $I = \mathfrak{m}$ and J is generated by monomials of the same degree [9]. Our results may give some hints for such a formula since the computation of the function $\ell(I^n J^m / I^{n+1} J^m)$ can be reduced to the computation of a Bhattacharya function in the two variables case.

The paper is organized as follows. In Section 1 we describe the decomposition of complete monomial ideals. In Section 2 we show how to compute the colength of an \mathfrak{m} -primary complete monomial ideal. Explicit formulas for the Bhattacharya function and their consequences are given in Section 3.

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1. DECOMPOSITION OF COMPLETE IDEALS

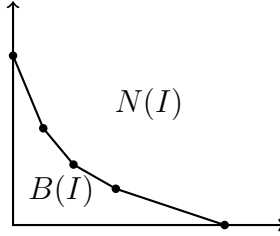
Let R be a Noetherian ring and I an ideal of R . An element $x \in R$ is said to be *integral* over I if it satisfies an equation of the form

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0$$

where $c_j \in I^j$ for $j = 1, \dots, n$. It is known that the set of all integral elements over I form an ideal of R . This ideal is called the *integral closure* of I , denoted by \bar{I} . If $\bar{I} = I$, then I is called an *integrally closed* or *complete* ideal. For more information on integrally closures of ideals we refer to the books [5] and [12].

The complete ideals behave especially well if R is a two-dimensional regular local ring. In this case we know by Zariski [13, Appendix 5] that the product of two complete ideals is complete and every complete ideal can be uniquely written as a product of simple complete ideals, where a complete ideal is *simple* if it is not the product of two complete ideals.

If I is a monomial ideal in a polynomial ring $R = k[x_1, \dots, x_n]$, the integral closure \bar{I} can be described combinatorially as follows. Let $N(I)$ denote the *Newton polyhedron* of I , that is the convex hull of the exponent vectors of the monomials of I in \mathbb{R}^n . Then \bar{I} is the ideal generated by all monomials whose exponent vectors belong to $N(I)$ (see [5] or [12]). Let $B(I)$ denote the union of the compact faces of $N(I)$. If we define a partial order on \mathbb{R}^n by the rule $\mathbf{a} \leq \mathbf{b}$ if each component of \mathbf{a} is less or equal than the corresponding component of \mathbf{b} , then $N(I)$ is the set of all points which are greater or equal the points of $B(I)$. So I is completely determined by $B(I)$. We call $B(I)$ the *Newton boundary* of I .



If $R = k[x, y]$ is a polynomial ring in two variables, the homogeneous version of Zariski's decomposition theorem implies that the product of two complete homogeneous ideals in R is complete and that every complete homogeneous ideal can be uniquely written as a product of simple homogeneous ideals. Let I be a complete monomial ideal in $R = k[x, y]$. One may ask whether there is a description of the simple homogeneous ideals of I in terms of $B(I)$.

The answer is yes and is due to Crispin Quinonez [2]. She calls the integral closure of a complete intersection ideal (x^p, y^q) with $\gcd(p, q) = 1$ a *block ideal* and proved that a block ideal is not the product of two complete monomial ideals [2, Proposition 3.4]. Actually, one can prove more.

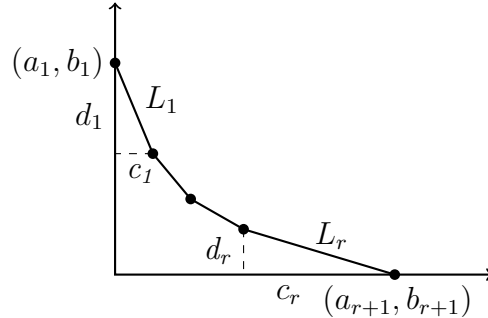
Proposition 1.1. *A block ideal is simple.*

Proof. Let $I = \overline{(x^p, y^q)}$ with $\gcd(p, q) = 1$. Then $B(I)$ is the line segment connecting the points $(p, 0)$ and $(0, q)$ and they are the only lattice points of $B(I)$. Since $mq + np = pq$ is the equation of the supporting line of this segment, all lattice points $(m, n) \in N(I)$ satisfy the condition $mq + np \geq pq$. Consider R as a weighted graded ring with $\deg x = q, \deg y = p$. By the above description of $N(I)$, all monomials of I have degree $\geq pq$ and x^p, y^q are the only monomials having degree pq .

Assume that I is the product of two complete ideals I_1, I_2 . For every polynomial f we denote by $\text{in}(f)$ the initial monomial of f with respect to the weighted degree. Let $f_1 \in I_1$ and $f_2 \in I_2$ such that $\deg(\text{in}(f_1))$ resp. $\deg(\text{in}(f_2))$ are minimal among the degree of the initial terms of the polynomials of I_1 resp. I_2 . Since $f_1 f_2 \in I$ and I is a monomial ideal, $\text{in}(f_1) \text{in}(f_2) \in I$ and $\deg(\text{in}(f_1) \text{in}(f_2)) = pq$, the smallest degree of the monomials of I . Therefore, $\text{in}(f_1) \text{in}(f_2)$ equals x^p or y^q . Without restriction we may assume that $\text{in}(f_1) \text{in}(f_2) = x^p$. Then $\text{in}(f_1) = x^a$ and $\text{in}(f_2) = x^b$ for some fixed non-negative integers a, b with $a + b = p$. From this it follows that for all $g_1 \in I_1$ resp. $g_2 \in I_2$, we either have $\text{in}(g_1) = x^a$ or $\deg(\text{in}(g_1)) > \deg(\text{in}(f_1)) = aq$ resp. $\text{in}(g_2) = x^b$ or $\deg(\text{in}(g_2)) > \deg(\text{in}(f_2)) = bq$. Thus, $\text{in}(g_1 g_2) = x^a x^b = x^p$ or $\deg(\text{in}(g_1 g_2)) > \deg(\text{in}(f_1)) + \deg(\text{in}(f_2)) = pq$. Since $\deg(y^q) = pq$, this implies $y^q \notin I_1 I_2 = I$, a contradiction. \square

Let \mathfrak{m} be the maximal homogeneous ideal of $R = k[x, y]$. Since every monomial ideal in R is the product of a monomial with an \mathfrak{m} -primary monomial ideal, we may assume that I is \mathfrak{m} -primary. In this case, Crispin Quinonez proved that I can be uniquely decomposed as a product of block ideals.

Her proof also describes the block ideals of I from the vertices of the Newton boundary $B(I)$. Let $(a_1, b_1), \dots, (a_{r+1}, b_{r+1})$ be the vertices of $N(I)$. Without loss of generality we may assume that $0 = a_1 < \dots < a_{r+1}$, which implies $b_1 > \dots > b_{r+1} = 0$. Let L_i denote the line segment connecting (a_i, b_i) to (a_{i+1}, b_{i+1}) , $i = 1, \dots, r$. Then $B(I) = L_1 \cup \dots \cup L_r$.



Put $c_i = a_{i+1} - a_i$ and $d_i = b_i - b_{i+1}$, $i = 1, \dots, r$. Let p_i, q_i be positive numbers with $\gcd(p_i, q_i) = 1$ such that $p_i/q_i = d_i/c_i$. Let $C_i = (x^{p_i}, y^{q_i})$ and $n_i = \gcd(c_i, d_i)$.

Theorem 1.2. ([2, Theorem 3.8]) *Let I be an \mathfrak{m} -primary complete monomial ideal in R . With the above notations we have*

$$I = C_1^{n_1} \dots C_r^{n_r}.$$

Geometrically, the slope of L_i is given by the ratio d_i/c_i and $n_i + 1$ is the number of lattice points on L_i . So we can decompose L_i into n_i line segments whose interior does not contain any lattice point. Since $B(C_i)$ is the line segment connecting $(p_i, 0)$ to $(0, q_i)$ which has the same slope, we may consider L_i as the union of n_i copy of $B(C_i)$. So we can read off the block ideals of I from $B(I)$.

Remark 1.3. It is sometime more convenient to write I as a product of integral closures of complete intersections which are not necessarily block ideals. Let $I = J_1 \cdots J_r$, where $J_i = \overline{(x^{c_i}, y^{d_i})}$ for some positive integers c_i, d_i , $i = 1, \dots, r$. Define p_i, q_i , and n_i as above. Then $I = C_1^{n_1} \cdots C_r^{n_r}$ is a product of block ideals, where C_1, \dots, C_r are not necessarily different. Without restriction we may assume that $d_1/c_1 \geq \cdots \geq d_r/c_r$. Define $a_1 = 0$, $b_{r+1} = 0$ and

$$\begin{aligned} a_i &= c_1 + \cdots + c_{i-1}, \quad i = 2, \dots, r+1, \\ b_i &= d_i + \cdots + d_r, \quad i = 1, \dots, r. \end{aligned}$$

Then we still have $B(I) = L_1 \cup \cdots \cup L_r$, although the points $(a_1, b_1), \dots, (a_{r+1}, b_{r+1})$ need not to be the vertices of $N(I)$.

For any subset $P \subset \mathbb{R}_+^2$ we denote by l_P the number of lattice points in P . For short we set $l_I = l_{B(I)}$.

Lemma 1.4. *Let $I = J_1 \cdots J_r$, where $J_i = \overline{(x^{c_i}, y^{d_i})}$ for some positive integers c_i, d_i , $i = 1, \dots, r$. Then*

$$l_I = \sum_{i=1}^r \gcd(c_i, d_i) + 1,$$

and $l_I - 1$ is the number of block ideals in the decomposition of I .

Proof. By Remark 1.3, $n_1 + \cdots + n_r$ is the number of block ideals in the decomposition of I . On the other hand, $n_i + 1$ is the number of lattice points on L_i , $i = 1, \dots, r$. Hence $n_1 + \cdots + n_r + r$ is the sum of the numbers of lattice points on L_1, \dots, L_r . Since $B(I) = L_1 \cup \cdots \cup L_r$ and since the points $(a_2, b_2), \dots, (a_r, b_r)$ are counted twice in the above sum, we get

$$l_I = (n_1 + \cdots + n_r + r) - (r - 1) = n_1 + \cdots + n_r + 1 = \sum_{i=1}^r \gcd(c_i, d_i) + 1.$$

□

Lemma 1.5. *Let $I = J_1 \cdots J_r$, where J_1, \dots, J_r are \mathfrak{m} -primary complete monomial ideals in R . Then*

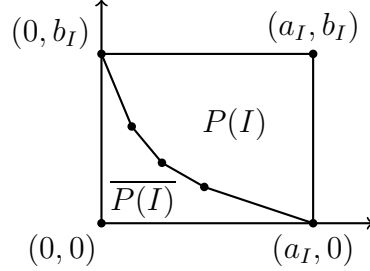
$$l_{J_1 \cdots J_r} = l_{J_1} + \cdots + l_{J_r} - r + 1.$$

Proof. By Theorem 1.2, the number of block ideals in the decomposition of $J_1 \cdots J_r$ is the sum of the numbers of block ideals in the decompositions of J_1, \dots, J_r . Hence the assertion follows from Lemma 1.4. □

2. COLENGTH OF AN COMPLETE MONOMIAL IDEAL

Let R be a polynomial ring over a field k . Let I be an \mathfrak{m} -primary complete monomial ideal in R , where \mathfrak{m} is the maximal homogeneous ideal of R . Then R/I is an R -module of finite length. One calls $\ell(R/I)$ the *colength* of I . It is clear that $\ell(R/I)$ is the number of the monomials not in I . Since I is generated by the monomials whose exponent vectors belong to $N(I)$, the monomials not in I correspond to the lattice points of the complement of $N(I)$ in \mathbb{N}^n .

Let I be an \mathfrak{m} -primary complete monomial ideal in $R = k[x, y]$. Let $(a_I, 0), (0, b_I)$ be the end points of the Newton boundary $B(I)$. Let $\overline{P(I)}$ be the polygon confined by $B(I)$ and the two line segments connecting the origin $(0, 0)$ with the points $(a_I, 0), (0, b_I)$. Then $\overline{P(I)} \setminus B(I)$ is the complement of $N(I)$ in \mathbb{N}^n . Hence $\ell(R/I) = l_{\overline{P(I)}} - l_I$.



We can estimate $l_{\overline{P(I)}}$ by using Pick's theorem which relates the lattice points of a polygon with its area. Recall that a polygon is a *lattice polygon* if its vertices are lattice points.

Let $V(P)$ denote the area of a lattice polygon P . Let i_P resp. b_P be the numbers of lattice points in the interior resp. the boundary of P .

Theorem 2.1. (Pick's Theorem, see e.g. [4]) $V(P) = i_P + \frac{b_P}{2} - 1$.

In the following we set $s_I := V(\overline{P(I)})$.

Lemma 2.2. $\ell(R/I) = s_I + \frac{1}{2}(a_I + b_I - l_I + 1)$.

Proof. By Theorem 2.1 we have

$$l_{\overline{P(I)}} = l_{\overline{P(I)}} - l_I = i_{\overline{P(I)}} + \frac{b_{\overline{P(I)}}}{2} - 1 + l_I = s_I + \frac{b_{\overline{P(I)}}}{2} + 1 - l_I.$$

It is easy to see that $b_{\overline{P(I)}} = a_I + b_I + l_I - 1$. Therefore,

$$\ell(R/I) = s_I + \frac{1}{2}(a_I + b_I + l_I - 1) + 1 - l_I = s_I + \frac{1}{2}(a_I + b_I - l_I + 1).$$

□

Remark 2.3. In general, $\overline{P(I)}$ is not a convex polygon. However, one can reduce the computation of s_I to the computation of volumes of convex polygons. Let $Q(I)$ denote the rectangle with the vertices $(0, 0), (a_I, 0), (b_I, 0), (a_I, b_I)$. Let $P(I)$ denote the convex polygon defined by $B(I)$ and the two line segments connecting $(a_I, 0), (0, b_I)$ with (a_I, b_I) . Then

$$(*) \quad s_I = V(Q(I)) - V(P(I)).$$

If we know a decomposition of I as a product of integral closures of complete intersections, we can compute s_I directly.

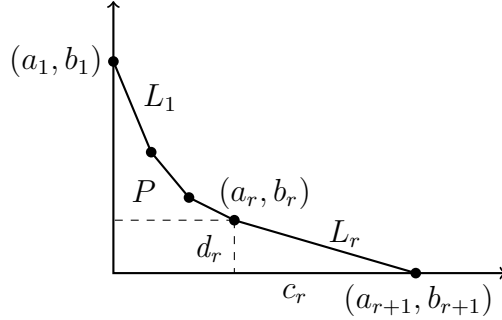
Lemma 2.4. Let $I = J_1 \cdots J_r$, where $J_i = \overline{(x^{c_i}, y^{d_i})}$ for some positive integers c_i, d_i , $i = 1, \dots, r$. Assume that $d_1/c_1 \geq \cdots \geq d_r/c_r$. Then

$$s_I = \sum_{i=1}^r \frac{c_i d_i}{2} + \sum_{1 \leq i < j \leq r} c_i d_j.$$

Proof. We proceed by induction. If $r = 1$, the assertion is trivial. If $r > 1$, let $J = J_1 \cdots J_{r-1}$. By the induction hypothesis,

$$s_J = \sum_{i=1}^{r-1} \frac{c_i d_i}{2} + \sum_{1 \leq i < j \leq r-1} c_i d_j.$$

Define a_i, b_i as in Remark 1.3, $i = 1, \dots, r+1$. Then $B(I) = L_1 \cup \cdots \cup L_r$, where L_i is the line segment connecting (a_i, b_i) to (a_{i+1}, b_{i+1}) , $i = 1, \dots, r$. It is clear that $\overline{P(J)}$ is a translation of the polygon P confined by $B(I)$, the vertical axis, and the horizontal line passing through the point (a_r, b_r) :



Thus, $s_I - s_J = V(\overline{P(I)} \setminus P) = \frac{c_r d_r}{2} + (c_1 + \cdots + c_{r-1})d_r$. Hence the assertion is immediate. \square

Now we can deduce an explicit formula for the colength of I in terms of the decomposition of I as a product of block ideals.

Theorem 2.5. Let $I = C_1^{n_1} \cdots C_r^{n_r}$ be the decomposition of I as a product of block ideals $C_i = \overline{(x^{p_i}, y^{q_i})}$, $i = 1, \dots, r$. Assume that $q_1/p_1 > \cdots > q_r/p_r$. Then

$$\ell(R/I) = \sum_{i=1}^r \frac{p_i q_i}{2} n_i^2 + \sum_{1 \leq i < j \leq r} p_i q_j n_i n_j + \frac{1}{2} \sum_{i=1}^r (p_i + q_i - 1) n_i.$$

Proof. Let $J_i = \overline{(x^{p_i n_i}, y^{q_i n_i})}$, $i = 1, \dots, r$. Then $J_i = C_i^{n_i}$. Hence $I = J_1 \cdots J_r$. By Lemma 2.4 we have

$$s_I = \sum_{i=1}^r \frac{p_i q_i}{2} n_i^2 + \sum_{1 \leq i < j \leq r} p_i q_j n_i n_j.$$

It is clear that $a_I = p_1 n_1 + \cdots + p_r n_r$ and $b_I = q_1 n_1 + \cdots + q_r n_r$. By Lemma 1.4, $l_I = n_1 + \cdots + n_r + 1$. If we put these data into Lemma 2.2, we obtain the assertion. \square

3. BHATTACHARYA FUNCTION

Let (R, \mathfrak{m}) be a d -dimensional local ring. Let I, J be two \mathfrak{m} -primary ideals. Recall that the function $\ell(R/I^m J^n)$ is a polynomial $P(m, n)$ of total degree d for $m, n \gg 0$ and that $\ell(R/I^m J^n)$ and $P(m, n)$ are called the *Bhattacharya function* and the *Bhattacharya polynomial* of I, J . If we write

$$P(m, n) = \sum_{0 \leq i, j \leq d} e_{i,j} \binom{m}{i} \binom{n}{j},$$

the numbers $e_{i,j}$ with $i + j = d$ are called the *mixed multiplicities* of I, J .

In this section we will deal with the case $R = k[x, y]$ is a polynomial ring of two variables over a field k and I, J are two \mathfrak{m} -primary complete monomial ideals, where \mathfrak{m} is the maximal homogeneous ideal. We will give an explicit formula for $\ell(R/I^m J^n)$ in combinatorial terms of I and J . For that we shall need the following notations.

Given two (not necessarily different) convex polygons P_1 and P_2 , the *Minkowski sum* of P_1, P_2 is defined by

$$P_1 + P_2 = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in P_1, \mathbf{b} \in P_2\}.$$

It is easy to see that $P_1 + P_2$ is again a convex polygon. The *mixed area* $MV(P_1, P_2)$ of P_1 and P_2 is defined by

$$MV(P_1, P_2) := V(P_1 + P_2) - V(P_1) - V(P_2).$$

Let mP_1 and nP_2 denote the Minkowski sums of n times P_1 and n times P_2 , respectively. It is a classical result that the area of $mP_1 + nP_2$ is a homogeneous polynomial in m, n which involves the mixed area of P_1 and P_2 .

Lemma 3.1. (see e.g. [3, Ch. 7, Proposition 4.9]) *Let P_1 and P_2 be convex polygons in \mathbb{R}^2 . Then*

$$V(mP_1 + nP_2) = V(P_1)m^2 + V(P_2)n^2 + MV(P_1, P_2)mn.$$

Lemma 3.2. *Let I and J be \mathfrak{m} -primary complete monomial ideals in R . Then*

- (i) $Q(I^m J^n) = mQ(I) + nQ(J)$,
- (ii) $P(I^m J^n) = mP(I) + nP(J)$.

Proof. We only need to prove the case $m = n = 1$ because this case can be applied successively to obtain the assertion. The formula $Q(IJ) = Q(I) + Q(J)$ follows from the definition of the Minkowski sum and the facts that $a_{IJ} = a_I + a_J$ and $b_{IJ} = b_I + b_J$. By the definition of the Newton polyhedron we have $N(IJ) = N(I) + N(J)$. Since the boundary of a Minkowski sum is contained in the Minkowski sum of the boundary of the summands, $B(IJ) \subseteq B(I) + B(J)$. From this it follows that $P(IJ) \subseteq P(I) + P(J)$. On the other hand,

$$P(I) + P(J) \subseteq (Q(I) + Q(J)) \cap (N(I) + N(J)) = Q(IJ) \cap N(IJ) = P(IJ).$$

Therefore, $P(IJ) = P(I) + P(J)$. □

Theorem 3.3. *Let I, J be \mathfrak{m} -primary complete monomial ideals in $R = k[x, y]$. For all $m, n \geq 0$,*

$$\begin{aligned}\ell(R/I^m J^n) &= s_I m^2 + s_J n^2 + (s_{IJ} - s_I - s_J)mn \\ &\quad + \frac{1}{2}(a_I + b_I - l_I + 1)m + \frac{1}{2}(a_J + b_J - l_J + 1)n.\end{aligned}$$

Proof. We will use Lemma 2.2 to compute $\ell(R/I^m J^n)$. By Remark 2.3 (*) we have

$$s_{I^m J^n} = V(Q(I^m J^n)) - V(P(I^m J^n)).$$

Using Lemma 3.1 and Lemma 3.2 we obtain a formula for $V(Q(I^m J^n)) - V(P(I^m J^n))$ in terms of the areas and mixed areas of $Q(I), Q(J)$ and $P(I), P(J)$. From this formula and Remark 1.3 (*) (applied to the ideals IJ, I, J) we can deduce that

$$s_{I^m J^n} = s_I m^2 + s_J n^2 + (s_{IJ} - s_I - s_J)mn.$$

It is clear that $a_{I^m J^n} = a_I m + a_J n$ and $b_{I^m J^n} = b_I m + b_J n$. By Lemma 1.5, $l_{I^m J^n} = (l_I - 1)m + (l_J - 1)n + 1$. Hence the assertion follows from Lemma 2.2. \square

From Theorem 3.3 we obtain the following formulas for the mixed multiplicities of I, J : $e_{2,0} = 2s_I$, $e_{0,2} = 2s_J$, $e_{1,1} = s_{IJ} - s_I - s_J$. These formulas can be also deduced from a more general result of Teissier [8, Corollary 8.7].

If (R, \mathfrak{m}) is a two-dimensional regular local ring, Verma [11, Corollary 3.5] showed that

$$\begin{aligned}\ell(R/I^m J^n) &= e(I) \binom{m}{2} + e(J) \binom{n}{2} + (\ell(R/IJ) - \ell(R/I) - \ell(R/J))mn \\ &\quad + \ell(R/I)m + \ell(R/J)n\end{aligned}$$

for all $m, n \geq 0$, where $e(I)$ and $e(J)$ denote the multiplicities of I and J . To prove this result he used the theory of joint reductions of complete ideals and Lipman's formula $e_{1,1} = \ell(R/IJ) - \ell(R/I) - \ell(R/J)$ [7]. By Teissier's result one has $e(I) = 2s_I$ and $e(J) = 2s_J$. If we use Lemma 2.2 and Lemma 1.5 to compute $\ell(R/I), \ell(R/J)$ and $\ell(R/IJ)$ we can also recover Theorem 3.3 from Verma's result.

As we have seen in the previous sections, the coefficients of the Bhattacharya polynomial in Theorem 3.3 can be written down explicitly if we know the vertices of the Newton polyhedra of I, J or the decompositions of I, J as products of integral closures of complete intersections. To see that we consider the case $J = \mathfrak{m}$.

Theorem 3.4. *Let $I = J_1 \cdots J_s$, where $J_i = \overline{(x^{c_i}, y^{d_i})}$ for some positive integers c_i, d_i , $i = 1, \dots, s$. Assume that $d_1/c_1 \geq \dots \geq d_s/c_s$. Let $s = \max\{i \mid d_i/c_i \geq 1\}$, where $s = 0$ if $d_1/c_1 < 1$. For all $m, n \geq 0$,*

$$\begin{aligned}\ell(R/I^m \mathfrak{m}^n) &= \left(\sum_{i=1}^s \frac{c_i d_i}{2} + \sum_{1 \leq i < j \leq s} c_i d_j \right) m^2 + \frac{n^2}{2} + \left(\sum_{i=1}^s c_i + \sum_{j=s+1}^r d_j \right) mn \\ &\quad + \frac{1}{2} \sum_{i=1}^s (c_i + d_i - \gcd(c_i, d_i)) m + \frac{n}{2}.\end{aligned}$$

Proof. By Lemma 2.4, s_I is the coefficient of m^2 in the right hand side of the above formula. It is clear that $s_{\mathfrak{m}} = 1/2$. Since $d_1/c_1 \geq \cdots \geq d_s/c_s \geq 1/1 > d_{s+1}/c_{s+1} \geq \cdots \geq d_r/c_r$, applying Lemma 2.4 to the ideal $I\mathfrak{m}$ we get

$$s_{I\mathfrak{m}} = \sum_{i=1}^r \frac{c_i d_i}{2} + \frac{1}{2} + \sum_{1 \leq i < j \leq r} c_i d_j + \sum_{i=1}^s c_i + \sum_{j=s+1}^r d_j = s_I + \frac{1}{2} + \sum_{i=1}^s c_i + \sum_{j=s+1}^r d_j.$$

By definition, $a_I = \sum_{i=1}^r c_i$, $b_I = \sum_{i=1}^r d_i$, and by Lemma 1.4, $l_I = \sum_{i=1}^r \gcd(c_i, d_i) + 1$. Moreover, $a_{\mathfrak{m}} = b_{\mathfrak{m}} = 1$, $l_{\mathfrak{m}} = 2$. Putting these data into Theorem 3.3 we obtain the assertion. \square

Theorem 3.4 contains explicit formulas for the Hilbert-function of the associated graded ring $G(I) = \bigoplus_{m \geq 0} I^m/I^{m+1}$ and the fiber ring $F(I) = \bigoplus_{m \geq 0} I^m/\mathfrak{m}I^m$.

Corollary 3.5. *Let $I = J_1 \cdots J_s$, where $J_i = \overline{(x^{c_i}, y^{d_i})}$ for some positive integers c_i, d_i , $i = 1, \dots, r$. Assume that $d_1/c_1 \geq \cdots \geq d_r/c_r$. For all $m \geq 0$,*

$$\ell(I^m/I^{m+1}) = \left(\sum_{i=1}^r \frac{c_i d_i}{2} + \sum_{1 \leq i < j \leq r} c_i d_j \right) (2m+1) + \frac{1}{2} \sum_{i=1}^r (c_i + d_i - \gcd(c_i, d_i)).$$

Proof. Since $\ell(I^m/I^{m+1}) = \ell(R/I^{m+1}) - \ell(R/I^m)$, the assertion follows from Theorem 3.4 by putting $n = 0$. \square

Corollary 3.6. *Let $I = J_1 \cdots J_s$, where $J_i = \overline{(x^{c_i}, y^{d_i})}$ for some positive integers c_i, d_i , $i = 1, \dots, r$. Assume that $d_1/c_1 \geq \cdots \geq d_r/c_r$. Let $s = \max\{i \mid d_i/c_i \geq 1\}$, where $s = 0$ if $d_1/c_1 < 1$. For all $m \geq 0$,*

$$\ell(I^m/\mathfrak{m}I^m) = \left(\sum_{i=1}^s c_i + \sum_{j=s+1}^r d_j \right) m + 1.$$

Proof. Since $\ell(I^m/\mathfrak{m}I^m) = \ell(R/\mathfrak{m}I^m) - \ell(R/I^m)$, the assertion follows from Theorem 3.4 by putting $n = 0, 1$. \square

In particular, the case $m = 1$ of Corollary 3.6 yields the following formula for the minimal number of generators $v(I)$ of the ideal I .

Corollary 3.7. *Let I be a complete monomial ideal as above. Then*

$$v(I) = \sum_{i=1}^s c_i + \sum_{j=s+1}^r d_j + 1.$$

We can also apply Theorem 3.3 to study the function $\ell(I^m J^n/I^{m+1} J^n)$ for an \mathfrak{m} -primary complete monomial ideal I and an arbitrary complete monomial ideal J in R . This follows from the fact that $J = fJ'$ for a monomial f and an \mathfrak{m} -primary complete monomial ideal J' . Hence

$$\ell(I^m J^n/I^{m+1} J^n) = \ell(I^m (J')^n/I^{m+1} (J')^n) = \ell(R/I^{m+1} (J')^n) - \ell(R/I^m (J')^n).$$

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